

Understand these concepts from Chapters 5 and 6:

1. Identify whether a set of vectors forms a basis for a set.
2. Know that the dimension of a vector space is equal to the number of vectors in its basis.
3. Know what the rank and nullity of a matrix are.
4. What eigenvalues and eigenvectors represent. Solutions to $Ax = \lambda x$
5. How to calculate the eigenvalues of a small matrix (2×2 or 3×3) by hand.
6. How to calculate the eigenvectors of a matrix.
7. What the algebraic multiplicity of an eigenvalue is.
8. Chapter 5, Theorem 2: If $\{v_1, v_2, \dots, v_r\}$ are eigenvectors which correspond to distinct eigenvalues, then the set $\{v_1, v_2, \dots, v_r\}$ is linearly independent.
9. How to write the characteristic equation of a matrix and what it is used for.
10. How to diagonalize a matrix. When would diagonalization be advantageous?
11. How to add, subtract, and multiply complex numbers.
12. How to find the complex eigenvalues and eigenvectors of a matrix.
13. Compute the dot product of a pair of vectors.
14. Compute the norm of a vector.
15. Determine the angle between a pair of vectors.
16. Identify orthogonal vectors and sets.
17. Find the projection of a vector onto a subspace.
18. Find the shortest distance from a vector to subspace.
19. Given an orthogonal basis $\{u_1, u_2, \dots, u_r\}$ for a subspace W write vector $y \in W$ as $y = c_1u_1 + c_2u_2 + \dots + c_ru_r$
20. Use the Gram-Schmidt process to construct an orthonormal basis for subspace W .

Practice Problems

1. Identify whether the following statements are true or false. If a statement is false, give an explanation why.

- (a) An $n \times n$ matrix has n eigenvalues.
- (b) $\frac{6 - 12i}{2 + 3i} = 3 - 4i$
- (c) If A is similar to the identity matrix, then $\det A = 0$.
- (d) If x is an eigenvector of A , then the line through the origin and x passes through Ax .
- (e) If B and C are bases for the same vector space V , then B and C contain the same number of vectors.
- (f) If u and v are in \mathbb{R}^n , then $u \cdot v = v \cdot u$.
- (g) A is a diagonalizable matrix if $A = PDP^{-1}$ for some matrix D and some invertible matrix P .

2. You are given that

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, P = [x_1 \ x_2 \ x_3], D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

and $A = PDP^{-1}$. Then $A^2x_2 =$

- (a) $\begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix}$
- (b) $\begin{bmatrix} 25 \\ 0 \\ 100 \end{bmatrix}$
- (c) $\begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix}$
- (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- (e) $\begin{bmatrix} 5 \\ 0 \\ 20 \end{bmatrix}$

3. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -4 & 6 \\ -1 & 1 \end{bmatrix}$

4. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -3 & 4 \\ 3 & 8 \end{bmatrix}$

5. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -5 \\ 9 & 3 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 9 & 4 \\ -1 & -3 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 8$. Find an eigenbasis for each eigenvalue.

7. The 3×3 matrix A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -4$, $\lambda_3 = 1$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Write A in the form PDP^{-1} , where D is a diagonal matrix.

8. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. Which of the following vectors is orthogonal to the column space of A ?

(a) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

(e) Since the second row is a multiple of the third row, $\text{Col } A$ is undefined.

9. Let $A = \begin{bmatrix} 4 & 2 & -3 \\ 3 & 4 & 1 \\ 4 & 1 & 5 \end{bmatrix}$. Then $\lambda = 3$ is an eigenvalue corresponding to the eigenvector $v = \begin{bmatrix} 1 \\ -2 \\ a \end{bmatrix}$. Find the value of a .

10. Let u, v be vectors in \mathbb{R}^n with θ the angle between u and v . Then $\|u + v\|^2$ is equal to:

(a) $\|u\|^2\|v\|^2 \cos \theta$

(b) $\|u\|^2 + \|v\|^2$

(c) $\|u\|^2 - \|v\|^2$

(d) $\|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \cos \theta$

(e) $\|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta$

11. Let

$$A = \begin{bmatrix} 5 & -1 & 3 & -1 \\ 0 & 4 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find h so that the eigenspace corresponding to the eigenvalue $\lambda = 5$ is 2-dimensional.

12. Suppose the 2×2 matrix A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ with eigenvectors v_1 and v_2 , respectively. If $u = 5v_1 + v_2$, then A^2u is equal to

- (a) $25v_1 + v_2$
- (b) $25v_1 + 3v_2$
- (c) $80v_1 + 9v_2$
- (d) $100v_1 + 3v_2$
- (e) $400v_1 + 9v_2$

13. An $n \times n$ matrix B has characteristic polynomial $p(\lambda) = -\lambda(\lambda - 3)^3(\lambda - 2)^2(\lambda + 1)$. Which of the following statements is **FALSE**?

- (a) rank $B = 6$.
- (b) $\det(B) = 0$.
- (c) $\det(B^T B) = 0$.
- (d) B is invertible.
- (e) $n = 7$.

14. Given vectors $u = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ find $u \cdot v$.

15. Find a unit vector in the direction of $v = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

16. Find the distance between the two vectors. $u = (6, -12), v = (-12, 12)$

17. Determine whether the set of vectors is orthogonal.

$$\begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 20 \\ 0 \\ -20 \end{bmatrix}, \begin{bmatrix} -20 \\ -20 \\ -20 \end{bmatrix}$$

18. Find the orthogonal projection of $y = \begin{bmatrix} -24 \\ 10 \end{bmatrix}$ onto $u = \begin{bmatrix} 4 \\ 20 \end{bmatrix}$.

19. Let W be the subspace spanned by $u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Write $y = \begin{bmatrix} 19 \\ 3 \\ 11 \end{bmatrix}$ as the sum of a vector in W and a vector orthogonal to W . Find the closest point to y in the subspace W and the shortest distance.

Practice Problems Answers

1. Identify whether the following statements are true or false. If a statement is false, give an explanation why.

(a) True, including multiplicity.

(b) False.

(c) False, $\det A = 1$.

(d) True.

(e) True

(f) True.

(g) True.

2. $\begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix}$

3. $E_{\lambda=-2} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ and $E_{\lambda=-1} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

4. $E_{\lambda=9} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ and $E_{\lambda=-4} = \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$

5. $E_{\lambda=3+3i\sqrt{5}} = \left\{ \begin{bmatrix} i\sqrt{5} \\ 3 \end{bmatrix} \right\}$

$$6. E_{\lambda=3} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } E_{\lambda=8} = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$7. A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

$$8. \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$9. a = -1.$$

$$10. \|u\|^2 + \|v\|^2 + 2\|u\|^2\|v\|^2 \cos \theta$$

$$11. h = 3$$

$$12. 80v_1 + 9v_2$$

13. B is invertible is the False statement because $\lambda = 0$ can only be an eigenvalue of a singular matrix.

$$14. u \cdot v = 15.$$

$$15. \text{unit vector} = \begin{bmatrix} 2/\sqrt{14} \\ 3/\sqrt{14} \\ -1/\sqrt{14} \end{bmatrix}$$

$$16. 30$$

17. No

$$18. \text{proj}_u y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

$$19. y = \begin{bmatrix} 18 \\ 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}. \text{ Closest point } (18, 7, 10). \text{ Shortest distance } \sqrt{18}.$$