Understand these concepts from Chapters 2, 3 and 4:

1. What is a determinant?
2. Know what a cofactor is and how to use them to compute determinants.
3. How elementary row operations affect the determinant of a matrix (Theorem 3, p. 171).
4. How to combine row operations and cofactor expansion efficiently.
5. The connection between determinants and invertibility.
6. How to compute the area of a triangle (and other polygons) using determinants.
7. The equivalent conditions of the Invertible Matrix Theorem. (a) - (r)
8. The definition of a vector space.
9. The three criteria one has to check to see if a subset of $\mathbb{R}^n$ is a subspace.
10. Span$\{v_1, v_2, \ldots, v_p\}$ is the set of all linear combinations $c_1v_1 + c_2v_2 + \cdots + c_pv_p$.
11. The span of a set of vectors in $\mathbb{R}^n$ is always a subspace of $\mathbb{R}^n$.
12. Compute the null space of a matrix and express that null space as the span of a set of vectors.
13. Determine whether a vector is in the null space of a given matrix.
14. Compute the column space of a matrix and express that column space as the span of a set of vectors.
15. Understand that if A is the standard matrix of a linear transformation, Nul A is a subset of the domain, and Col A is the range.
16. Determine whether a set of vectors forms a basis for its span.
17. Compute a basis for Nul A and for Col A.
18. Know that the dimension of a vector space is equal to the number of vectors in its basis.
19. Know what the rank and nullity of a matrix are.
20. Use the rank of the matrix to answer questions like those in the left-hand column of page 239.
21. Given a basis $\mathcal{B}$, and a vector x, find $[x]_{\mathcal{B}}$.
22. Given a basis $\mathcal{B}$, and a vector $[x]_{\mathcal{B}}$, find $[x]$.
23. Given a matrix A, construct a basis for Row A (=Col A^T) and find its dimension.

Practice Problems

1. Calculate the area of the parallelogram formed between the points (0,3), (2,4), (5,2), and (3, 1).

2. The vector $\begin{bmatrix} a \\ b \\ 10 \\ 5 \end{bmatrix}$ is in the null space of $\begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 4 & 1 & 2 \end{bmatrix}$. Find the values of a and b.

3. (a) Calculate the determinant of

$$B = \begin{bmatrix} 0 & 4 & 0 & 1 & 0 \\ 9 & 1 & 0 & 11 & 3 \\ 3 & 6 & 0 & 8 & 0 \\ 2 & 5 & 4 & 1 & 7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

by using cofactor expansion efficiently.

(b) Now that you have the determinant of B, what can you say about the determinant of the matrix C shown here:

$$C = \begin{bmatrix} 9 & 1 & 0 & 11 & 3 \\ 0 & 12 & 0 & 3 & 0 \\ 3 & 6 & 0 & 8 & 0 \\ 2 & 5 & 4 & 1 & 7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

4. Suppose you knew that the columns of the 5×5 matrix A were linearly dependent. What can you say about the determinant of A?

5. Show that the set of vectors $\begin{bmatrix} 2r+3s \\ r-s \\ 5r \end{bmatrix}$ form a subspace of $\mathbb{R}^3$.

6. Show that the integer lattice $\mathbb{Z}^2$, which is the set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where x and y are whole integers, does not form a subspace of $\mathbb{R}^2$.

7. Suppose a 4×6 matrix A has rank 2. Then

   (a) Nul A is a _____-dimensional subspace of $\mathbb{R}^{\_\_}$.

   (b) Col A is a _____-dimensional subspace of $\mathbb{R}^{\_\_}$.

8. (a) What is the maximum rank of a 3×7 matrix?

   (b) The 4×9 matrix A has a rank of 3. What is its nullity?

9. If A is a 9×6 matrix with rank $A = 6$, what is the nullity of A?

10. If A is a 4×5 matrix that is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

what is the number of pivot positions that A has? What is the rank of A? What is the nullity of A? Can you name a basis for the row space of A? Why might the first, fourth, and fifth columns of B fail to form a basis for the column space of A?

11. If A is the matrix of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^7$, and A has exactly three vectors in the basis of its null space, what is the dimension of the row space of A?

12. If A is a 6×3 matrix, can A have a 4 dimensional row space? Can A have a 4 dimensional null space?

13. Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, so B is a basis for $\mathbb{R}^2$. Express the vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a coordinate vector relative to $\mathcal{B}$ (that is, find $[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$).

14. Let $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, so B is a basis for $\mathbb{R}^2$. Given the coordinates of x in this basis: $c_1 = 3,\ c_2 = 3$, what are the coordinates of x in the standard basis.

15. Let A be an $n \times n$ invertible matrix. Label the following as true or false:

    (a) $\dim \text{col } A^T = n$

    (b) If $A \sim B$, then B is invertible.

    (c) The rows of A are linearly independent and span $\mathbb{R}^n$.

    (d) The columns of A^T are linearly independent.

    (e) $\det A^T = \det A$.

    (f) If B contains exactly the same rows as A, but in a different order, then B is invertible.

    (g) The transformation $T(x) = Ax$ is both one-to-one and onto.

    (h) The equation $Ax = 0$ has an infinite number of solutions.

    (i) The reduced echelon form of A is an identity matrix.

    (j) Nul A is a single point.

16. If the row space of A is a two-dimensional subspace of $\mathbb{R}^3$, is it possible to determine the number of rows of A? How about the number of linearly independent rows of A?

17. If A is an $n \times n$ matrix and $A \sim I$, then do we know the rank of A^T?

18. Use a combination of row reduction and cofactor expansion to calculate det B where

$$B = \begin{bmatrix} 1 & 4 & 6 & 0 \\ 4 & 2 & 3 & 0 \\ 6 & 6 & 8 & 6 \\ 5 & 3 & 5 & 3 \end{bmatrix}$$

19. If $A = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 3 & 1 & 7 & 3 \end{bmatrix}$, find a basis for Nul A and Col A.

20. Mark each statement TRUE or, FALSE and why.

 (a) If A is a 2×2 matrix and det $A = 0$, then one column of A is a multiple of the other.

 (b) If A is a 3×3 matrix, then det $5A = 5$det A.

 (c) det $A^T A \geq 0$.

 (d) A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.

 (e) If $\{v_1, \ldots, v_n\}$ are vectors in a vector space V, then Span $\{v_1, \ldots, v_n\}$ is a subspace of V.

 (f) The set of pivot columns of a matrix is linearly independent.

 (g) If A is a 3×5 matrix, then Nul A is a subspace of $\mathbb{R}^5$.

 (h) If $\mathcal{B}$ and $\mathcal{C}$ are bases for the same vector space V, then $\mathcal{B}$ and $\mathcal{C}$ contain the same number of vectors.

 (i) If A is a 3×9 matrix in echelon form, then rank $A = 3$.

Practice Problems Answers

1. $\det \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} = -7$, so the area is $|-7| = 7$

2. $a = 8$ and $b = -7$.

3. (a) $\det B = 72$
   (b) $\det C = -216$, Switch two rows (negative) and multiply one row by 3 (3 times larger)

4. $\det A = 0$

5. $\begin{bmatrix} 2r + 3s \\ r - s \\ 5r \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$ and all spans are subspaces. Also satisfies closure, has the zero vector in it and has the standard multiplication and addition properties.

6. Let $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then x is in $\mathbb{Z}^2$, but $\frac{1}{2}x$ is not, so $\mathbb{Z}^2$ is not closed under scalar multiplication.

7. Suppose a 4×6 matrix A has rank 2. Then
   (a) Nul A is a 4-dimensional subspace of $\mathbb{R}^6$.
   (b) Col A is a 2-dimensional subspace of $\mathbb{R}^4$.

8. (a) 3
   (b) 6

9. 0

10. 3. 3. 2. Row $A = \{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \}$. Because you can only get zero in the last position from those columns.

11. A is 7×3, so it has 3 columns, and $3 - 3 = 0$ is the rank of A, so the dimension of the row space is 0. A is matrix with all of its entries equal to zero.

12. No because the maximum number of pivot rows is min(6,3). No because the dimension of the null space can't be larger than the number of columns.

13. $[x]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

14. $x = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

15. Let A be an $n \times n$ invertible matrix. Label the following as true or false:
   (a) true
   (b) true
   (c) true
   (d) true
   (e) true
   (f) true
   (g) true
   (h) false
   (i) true
   (j) true

16. Not enough information to determine the number of rows of A but we do know that there are 2 linearly independent rows in A. For example both of these matrices form 2-dimensional subspaces of $\mathbb{R}^3$. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

17. A^T is invertible so by the invertible matrix theorem it has rank n

18. det $B = 84$

19. Nul $A = \text{Span}\left\{\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

   Col $A = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}\right\}$.

20. Mark each statement TRUE or, FALSE and why.
   (a) TRUE
   (b) FALSE det $5A = 5^3$ det A.
   (c) TRUE
   (d) FALSE, doesn't necessarily contain zero vector
   (e) TRUE
   (f) TRUE
   (g) TRUE
   (h) TRUE
   (i) FALSE, could have row(s) of zeros. rank $A \leq 3$.